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Wave processes in the locally nonhomogeneous solids

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There is proposed a method of studying wave processes in locally nonhomogeneous solids with account for geometrically non-uniform surface. The method is based on the equation system of the locally nonhomogeneous elastic solid model obtained within the local gradient approach and the use of averaging operation to separate oscillatory and slowly variable over period of oscillation components of displacement and density fields. At the example of a layer there is illustrated an application of the method to study the frequencies of natural oscillations for different fixing conditions at the layer surfaces. It was established that the dependence of frequencies of natural oscillations of the layer on the characteristic sizes the nearsurface and structural nonhomogeneities in the case of the free layer surfaces is much higher comparing to the fixed surfaces case.

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1. Introduction

Nowadays technology employ extensively machines and devices built of elements that are heterogeneous in terms of their physical properties. The practice of engineering use of modern materials such as nanocomposites, nanostructured materials, characterized by a significant surface to volume ratio increases. These elements operate in various conditions including a vibration load. The issue of operational characteristics in such conditions is of considerable practical and theoretical interest.

A variety of mathematical models are used to describe and predict the behavior of the elements under different conditions of exploitation. The advanced approaches of continuum mechanics have to be applied to build an adequate model of the element with finite size effects, surface and interface effects. Among such approaches the approaches of nonlocal theory of elasticity [1,2] as well as local gradient approach in thermomechanics [3,4] are distinguished.

The mathematical models of local gradient approach are constructed using methods of irreversible thermodynamics and nonlinear mechanics. The introduction of reversible component of the mass flow vector allowed us to modify the mass balance equation for locally nonhomoheneous systems. Comparing to classical models the state parameter space is expanded with the density and the conjugate parameter chemical potential [3]. The key systems formulated for such approach were used for investigation of nearsurface nonhomogeneity and related phenomena in elastic, thermoelastic solids and solid solutions [6]. The attention was paid to studying the size effects of surface stresses, strength, surface tension etc. Further to coordinate the reference and actual states in the model the mass sources were introduced. The mass sources were related to the way of forming of the real surface of the body. In the model presentation they allow to consider geometric non-uniformity (roughness) of such surface [4,6].

In this paper the method of wave processes studying in deformable solids taking into account the structural nonhomogeneity of material and geometric non-uniformity of the surface is proposed. On the basis of the equations system of the model of locally nonhomogeneous elastic body, enriched with mass sources, there are studied the natural oscillations of locally nonhomogeneous elastic layer under different conditions at its surfaces. The dependence of the normal mode frequencies on the characteristic sizes of the nearsurface and structural nonhomogeneities is investigated.

2. The key system of nonhomogeneous elastic solid and the averaging operation

We consider key system of equations describing processes in locally nonhomogeneous elastic bodies [6]

$$\frac{\partial}{\partial \tau}\left(\rho \frac{\partial \boldsymbol{u}}{\partial \tau}\right) = \mu \nabla^2 \boldsymbol{u} + (\lambda+\mu)\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{u}\right) - (3\lambda+2\mu)a_m \boldsymbol{\nabla}\rho,$$

$$\nabla^2\rho - \xi^2(\rho-\rho_*) = -\xi^2 d_{\sigma m}. \tag{1}$$

Here $\boldsymbol{u}$ is the velocity vector, ρ and ρ_* are the densities in actual and reference state respectively, $d_{\sigma m}$ is the mass source related to body structure formation [4], τ is the time, μ, λ, a_m, ξ are material parameters. This system is nonlinear due to expression in the brackets in the left hand side of the first equation, this expression representing the momentum of mechanical motion. The mass sources must satisfy the condition

$$\int_{(V)} d_{\sigma m} dV = \int_{(V)} (\rho-\rho_*)\, dV,$$

where (V) is the region occupied by body.

We consider the locally nonhomogeneous body that is under the external action in the form of periodic power load. We assume that such action does not change parameters of nearsurface and structural nonhomogeneities. The solution of equation system (1) is represented as the sum of the oscillatory $\tilde{\boldsymbol{u}}, \tilde{\rho}$ and the slowly varying over time $\bar{\boldsymbol{u}}, \bar{\rho}$ components

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + \tilde{\boldsymbol{u}}, \qquad \rho = \bar{\rho} + \tilde{\rho}. \tag{2}$$

Such a presentation is made using averaging operation [7,8]

$$\bar{f}(\tau) = \frac{1}{\tau_0}\int_{\tau}^{\tau+\tau_0} f(t)\, dt. \tag{3}$$

Relation (3) yields

$$\overline{\frac{d^n f}{d\tau^n}} = \frac{d^n \bar{f}}{d\tau^n}. \tag{4}$$

We also accept approximations

$$\overline{\bar{f}\bar{\Phi}} \approx \bar{f}\bar{\Phi}, \qquad \overline{\bar{f}\tilde{\Phi}} \approx 0, \qquad \bar{\tilde{\Phi}} \approx 0 \tag{5}$$

of theory of nonlinear wave processes according to [7]. These ratios are held accurately when considering steady mode oscillations.

Given that the external action does not change the structure of the body we assume that oscillatory component of the density is absent, i.e. $\rho = \bar{\rho}$.

Using representation (2) in the system (1) we obtain

$$\frac{\partial}{\partial \tau}\left(\bar{\rho}\left(\frac{\partial \bar{\boldsymbol{u}}}{\partial \tau} + \frac{\partial \tilde{\boldsymbol{u}}}{\partial \tau}\right)\right) = \mu\nabla^2(\bar{\boldsymbol{u}}+\tilde{\boldsymbol{u}}) + (\lambda+\mu)\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot(\bar{\boldsymbol{u}}+\tilde{\boldsymbol{u}})\right) - (3\lambda+2\mu)a_m\boldsymbol{\nabla}\bar{\rho},$$

$$\nabla^2\bar{\rho} - \xi^2(\bar{\rho} - \rho_*) = -\xi^2\bar{d}_{\sigma m}. \tag{6}$$

Due to the smallness of averaged inertia force term from (6) we write such systems for averaged and oscillatory components of considered fields

$$\mu\nabla^2\bar{\boldsymbol{u}} + (\lambda+\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}) - (3\lambda+2\mu)a_m\boldsymbol{\nabla}\bar{\rho} = 0,$$
$$\nabla^2\bar{\rho} - \xi^2(\bar{\rho} - \rho_*) = -\xi^2\bar{d}_{\sigma m}; \tag{7}$$

$$\bar{\rho}\frac{\partial^2\tilde{\boldsymbol{u}}}{\partial\tau^2} = \mu\nabla^2\tilde{\boldsymbol{u}} + (\lambda+\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{u}}). \tag{8}$$

Thus the study of wave processes in locally non-homogeneous elastic bodies taking into account structural and nearsurface nonhomogeneities is reduced to a consecutive determination of averaged components on the basis of equations (7) with the following investigation of oscillatory component of the displacement vector from (8).

3. Structural and nearsurface nonhomogeneities effect on natural frequencies of the layer

Let us apply the formulated above system of equations to study the influence of parameters of structural nonhomogeneity of the material and geometric nonuniformity of the body surface on the natural oscillation frequency of the elastic layer that occupies domain $|x| \leqslant l$ of Euclidean space. Note that similar research in the case when the surface value of the chemical potential is prescribed and the mass sources are abcent was considered in [9,10].

To find fields

$$\bar{\rho} = \bar{\rho}(x), \qquad \tilde{\boldsymbol{u}} = (\tilde{u}_x(x,\tau), 0, 0)$$

we have the system of equations

$$\frac{d^2\bar{\rho}}{dx^2} - \xi^2(\bar{\rho} - \rho_*) = -\xi^2\bar{d}_{\sigma m},$$
$$\frac{\partial^2\tilde{u}_x}{\partial x^2} - \frac{1}{c_1^2}\frac{\bar{\rho}}{\rho_*}\frac{\partial^2\tilde{u}_x}{\partial\tau^2} = 0. \tag{9}$$

Here

$$c_1 = \sqrt{\frac{\lambda+2\mu}{\rho_*}}$$

is the speed of longitudinal wave propagation in a medium with parameters λ, μ, ρ_*.

We assume that both surfaces $x = \pm l$ of the layer are identical and the mass source is

$$\bar{d}_{\sigma m}(x) = m_s\frac{\cosh(\zeta x)}{\cosh(\zeta l)},$$

where the parameter m_s is choosen to satisfy relation

$$\int_{-l}^{l} m_s\frac{\cosh(\zeta x)}{\cosh(\zeta l)}dx = \int_{-l}^{l}(\bar{\rho} - \rho_*)dx. \tag{10}$$

The solution of Eq. (9) that satisfies boundary condition

$$\bar{\rho} = \rho_a \tag{11}$$

at surfaces $x = \pm l$ has the form

$$\bar{\rho} = \rho_* \left[1 - \frac{1 - r_{a*}}{1 - D}\left(\frac{\cosh(\xi x)}{\cosh(\xi l)} - D\frac{\cosh(\zeta x)}{\cosh(\zeta l)}\right)\right], \tag{12}$$

where

$$r_{a*} = \frac{\rho_a}{\rho_*}, \qquad D = \frac{\xi \tanh(\xi l)}{\zeta \tanh(\zeta l)}.$$

Using this solution we write second equation of (9) as

$$\frac{\partial^2 \tilde{u}_x}{\partial x^2} - \frac{1}{c_1^2}\left[1 - \frac{1 - r_{a*}}{1 - D}\left(\frac{\cosh(\xi x)}{\cosh(\xi l)} - D\frac{\cosh(\zeta x)}{\cosh(\zeta l)}\right)\right]\frac{\partial^2 \tilde{u}_x}{\partial \tau^2} = 0. \tag{13}$$

Considering periodic external load we seek solution of the equation in the form

$$\tilde{u}_x(x, \tau) = u(x)\exp(i\nu\tau). \tag{14}$$

The amplitude of oscillatory displacement (14) satisfies the equation

$$\frac{d^2 u}{dx^2} + k^2\left[1 - \frac{1 - r_{a*}}{1 - D}\left(\frac{\cosh(\xi x)}{\cosh(\xi l)} - D\frac{\cosh(\zeta x)}{\cosh(\zeta l)}\right)\right]u = 0. \tag{15}$$

Here $k = \nu/c_1$.

We accept $\zeta \geqslant \xi$. This does not reduce generality of further consideration because we may change notation for ξ, ζ and D. The Eq. (15) we rewrite as

$$\frac{d^2 u}{dx^2} + k^2\left[1 - \frac{\alpha}{1 - D}\left(\cosh(\xi x) - D\frac{\cosh(\xi l)}{\cosh(\zeta l)}\cosh(\zeta x)\right)\right]u = 0, \tag{16}$$

where

$$\alpha = \frac{1 - r_{a*}}{\cosh(\xi l)}.$$

Next we confine consideration to the layers the thickness of whose is much larger than the typical size of area of the structural and nearsurface non-homogeneities such that the following inequalities are hold

$$\exp(\xi l) \gg \xi l, \qquad \exp(\zeta l) \gg \zeta l.$$

The solution of (16) can be represented using expansion over a small parameter α

$$u(x) = u_0(x) + \alpha u_1(x) + \alpha^2 u_2(x) + \ldots \tag{17}$$

For the zero and first order approximations over α we obtain equations

$$\frac{d^2 u_0}{dx^2} + k^2 u_0 = 0,$$

$$\frac{d^2 u_1}{dx^2} + k^2 u_1 = \frac{k^2}{1 - D}\left(\cosh(\xi x) - D\frac{\cosh(\xi l)}{\cosh(\zeta l)}\cosh(\zeta x)\right)u_0. \tag{18}$$

The solution of (18) is

$$u_0 = A_0\cos(kx) + B_0\sin(kx),$$

$$u_1 = \frac{k^2}{1 - D}\left\{A_0\left[\frac{1}{\xi^2 + 4k^2}\left((\cosh(\xi x) - 1)\cos(kx) + 2\frac{k}{\xi}\sinh(\xi x)\sin(kx)\right) - \right.\right.$$

$$-D\frac{\cosh(\xi l)}{\cosh(\zeta l)}\frac{1}{\zeta^2+4k^2}\left((\cosh(\zeta x)-1)\cos(kx)+2\frac{k}{\zeta}\sinh(\zeta x)\sin(kx)\right)\Bigg]+$$

$$+B_0\left[\frac{1}{\xi^2+4k^2}\left((\cosh(\xi x)+1)\sin(kx)-2\frac{k}{\xi}\sinh(\xi x)\cos(kx)\right)-\right.$$

$$\left.-D\frac{\cosh(\xi l)}{\cosh(\zeta l)}\frac{1}{\zeta^2+4k^2}\left((\cosh(\zeta x)+1)\sin(kx)-2\frac{k}{\zeta}\sinh(\zeta x)\cos(kx)\right)\right]\Bigg\}. \quad (19)$$

Here A_0, B_0 are constants to be determined on the base of the properly stated boundary conditions.

Fixed surfaces of the layer. If the surfaces are fixed then the displacement is set to zero

$$u(-l)=0, \qquad u(l)=0. \quad (20)$$

For the first order approximation and accepted above assumptions from (17), (19) and (20) we write

$$A_0\left\{\cos(kl)+\frac{\alpha k^2}{1-D}\left[\frac{1}{\xi^2+4k^2}\left((\cosh(\xi l)-1)\cos(kl)+2\frac{k}{\xi}\sinh(\xi l)\sin(kl)\right)-\right.\right.$$

$$\left.\left.-D\frac{\cosh(\xi l)}{\cosh(\zeta l)}\frac{1}{\zeta^2+4k^2}\left((\cosh(\zeta l)-1)\cos(kl)+2\frac{k}{\zeta}\sinh(\zeta l)\sin(kl)\right)\right]\right\}+$$

$$+B_0\left\{\sin(kl)+\frac{\alpha k^2}{1-D}\left[\frac{1}{\xi^2+4k^2}\left((\cosh(\xi l)+1)\sin(kl)-2\frac{k}{\xi}\sinh(\xi l)\cos(kl)\right)-\right.\right.$$

$$\left.\left.-D\frac{\cosh(\xi l)}{\cosh(\zeta l)}\frac{1}{\zeta^2+4k^2}\left((\cosh(\zeta l)+1)\sin(kl)-2\frac{k}{\zeta}\sinh(\zeta l)\cos(kl)\right)\right]\right\}=0,$$

$$A_0\left\{\cos(kl)+\frac{\alpha k^2}{1-D}\left[\frac{1}{\xi^2+4k^2}\left((\cosh(\xi l)-1)\cos(kl)+2\frac{k}{\xi}\sinh(\xi l)\sin(kl)\right)-\right.\right.$$

$$\left.\left.-D\frac{\cosh(\xi l)}{\cosh(\zeta l)}\frac{1}{\zeta^2+4k^2}\left((\cosh(\zeta l)-1)\cos(kl)+2\frac{k}{\zeta}\sinh(\zeta l)\sin(kl)\right)\right]\right\}-$$

$$-B_0\left\{\sin(kl)+\frac{\alpha k^2}{1-D}\left[\frac{1}{\xi^2+4k^2}\left((\cosh(\xi l)+1)\sin(kl)-2\frac{k}{\xi}\sinh(\xi l)\cos(kl)\right)-\right.\right.$$

$$\left.\left.-D\frac{\cosh(\xi l)}{\cosh(\zeta l)}\frac{1}{\zeta^2+4k^2}\left((\cosh(\zeta l)+1)\sin(kl)-2\frac{k}{\zeta}\sinh(\zeta l)\cos(kl)\right)\right]\right\}=0, \quad (21)$$

A necessary condition for the existence of a nontrivial solution of the system of equations (21) is the zero determinant of the system. Confining ourselves to linear approximation and considering layers whose thickness is much larger than the characteristic sizes of the structural and nearsurface nonhomogeneities, we obtain

$$\left[1+\frac{k^2(1-r_{a*})}{1-D}\left(\frac{1}{\xi^2+4k^2}-\frac{D}{\zeta^2+4k^2}\right)\right]\sin(2kl)=$$

$$=\frac{k^2(1-r_{a*})}{1-D}\left(\frac{k}{\xi}\frac{1}{\xi^2+4k^2}-\frac{k}{\zeta}\frac{D}{\zeta^2+4k^2}\frac{\cosh(\xi l)}{\cosh(\zeta l)}\right)\cos(2kl). \quad (22)$$

If the structural and nearsurface nonhomogeneities are ommited from consideration then the analogous to (22) equation is

$$\sin(2kl)=0. \quad (23)$$

The last equation solution yields such wave numbers

$$k_0 = \frac{\pi n}{2l}, \quad n = 1, 2, 3, \ldots \tag{24}$$

Given that $k_0 = \nu/c_1$ we write the expression for natural frequencies of the layer with fixed surfaces

$$\nu_p = \frac{\pi n c_1}{2l}, \quad n = 1, 2, 3, \ldots \tag{25}$$

On this basis, one could argue that equation (22) is a transcendental equation for finding the natural frequencies of the layer taking into account the structural non-homogeneity of the material and geometric surface nonuniformity.

For the layers whose thickness is much larger than the specific size of structural and nearsurface nonhomogeneities we accept

$$k = k_0 + k_1, \quad k_1/k_0 \ll 1. \tag{26}$$

In the first approximation with respect to small parameter k_1/k_0 for parameter k from Eq. (22) we obtain

$$k = \frac{\pi n}{2l}\left[1 + (1 - r_{a*})\left(\frac{\pi n}{2}\right)^2\left(\frac{1}{(\xi l)^3} + \frac{1}{(\zeta l)^3}\right)\right], \quad n = 1, 2, 3, \ldots \tag{27}$$

This corresponds to such natural frequencies

$$\nu_n = \frac{\pi n c_1}{2l}\left[1 + (1 - r_{a*})\left(\frac{\pi n}{2}\right)^2\left(\frac{1}{(\xi l)^3} + \frac{1}{(\zeta l)^3}\right)\right], \quad n = 1, 2, 3, \ldots \tag{28}$$

The layer with fixed and free surfaces. If one surface is fixed and another surface is free then

$$u(-l) = 0, \qquad \left.\frac{du}{dx}\right|_{x=l} = 0. \tag{29}$$

In this case from above consideration we write such systems of linear algebraic equations

$$\begin{aligned}
&A_0\left\{\cos(kl) + \frac{\alpha k^2}{1-D}\left[\frac{1}{\xi^2+4k^2}\left((\cosh(\xi l) - 1)\cos(kl) + 2\frac{k}{\xi}\sinh(\xi l)\sin(kl)\right) -\right.\right. \\
&\qquad \left.\left. - D\frac{\cosh(\xi l)}{\cosh(\zeta l)}\frac{1}{\zeta^2+4k^2}\left((\cosh(\zeta l) - 1)\cos(kl) + 2\frac{k}{\zeta}\sinh(\zeta l)\sin(kl)\right)\right]\right\} - \\
&- B_0\left\{\sin(kl) + \frac{\alpha k^2}{1-D}\left[\frac{1}{\xi^2+4k^2}\left((\cosh(\xi l) + 1)\sin(kl) - 2\frac{k}{\xi}\sinh(\xi l)\cos(kl)\right) -\right.\right. \\
&\qquad \left.\left. - D\frac{\cosh(\xi l)}{\cosh(\zeta l)}\frac{1}{\zeta^2+4k^2}\left((\cosh(\zeta l) + 1)\sin(kl) - 2\frac{k}{\zeta}\sinh(\zeta l)\cos(kl)\right)\right]\right\} = 0, \\
&A_0\left\{-k\sin(kl) + \frac{\alpha k^2}{1-D}\left[\frac{1}{\xi^2+4k^2}\left(k\left(\cosh(\xi l) - 1\right)\sin(kl) + \frac{\xi^2+2k^2}{\xi}\sinh(\xi l)\cos(kl)\right) -\right.\right. \\
&\qquad \left.\left. - D\frac{\cosh(\xi l)}{\cosh(\zeta l)}\frac{1}{\zeta^2+4k^2}\left(k\left(\cosh(\zeta l) - 1\right)\sin(kl) + \frac{\zeta^2+2k^2}{\zeta}\sinh(\zeta l)\cos(kl)\right)\right]\right\} + \\
&+ B_0\left\{k\cos(kl) + \frac{\alpha k^2}{1-D}\left[\frac{1}{\xi^2+4k^2}\left(k\left(1 - \cosh(\xi l)\right)\cos(kl) + \frac{\xi^2+2k^2}{\xi}\sinh(\xi l)\sin(kl)\right) -\right.\right. \\
&\qquad \left.\left. - D\frac{\cosh(\xi l)}{\cosh(\zeta l)}\frac{1}{\zeta^2+4k^2}\left(k\left(1 - \cosh(\zeta l)\right)\cos(kl) + \frac{\zeta^2+2k^2}{\zeta}\sinh(\zeta l)\sin(kl)\right)\right]\right\} = 0.
\end{aligned} \tag{30}$$

Within the above assumptions for parameter k finding from condition of non-zero solution of (30) existence we get equation

$$\cot(2kl) + k\frac{1-r_{a*}}{1-D}\left(\frac{1}{\xi} - \frac{D}{\zeta}\right) = 0. \tag{31}$$

If effects of the nonhomogeneities are ommited from consideration then the equation corresponding to (31) is

$$\cos(2kl) = 0. \tag{32}$$

In the first approximation for parameter k we obtain

$$k = \frac{\pi(2n+1)}{2l}\left[1 + \frac{1}{2}\frac{1-r_{a*}}{1-D}\left(\frac{1}{\xi l} - \frac{D}{\zeta l}\right)\right], \quad n = 0, 1, 2, \ldots \tag{33}$$

Thus natural frequencies for the layer with one free surface are

$$\nu_n = \frac{\pi(2n+1)c_1}{2l}\left[1 + \frac{1}{2}\frac{1-r_{a*}}{1-D}\left(\frac{1}{\xi l} - \frac{D}{\zeta l}\right)\right], \quad n = 0, 1, 2, \ldots \tag{34}$$

The layer with free surfaces. In this case the boundary conditions for amplitude of wave component of displacement vector component are

$$\left.\frac{du}{dx}\right|_{x=-l} = 0, \qquad \left.\frac{du}{dx}\right|_{x=l} = 0. \tag{35}$$

Using the above procedure and approximations we obtain such expression for the natural frequencies for the layer with free surfaces

$$\nu_n = \frac{\pi(2n+1)c_1}{2l}\left[1 + \frac{1-r_{a*}}{1-D}\left(\frac{1}{\xi l} - \frac{D}{\zeta l}\right)\right], \quad n = 0, 1, 2, \ldots \tag{36}$$

From comparison of formulas (28), (34) and (36) of the natural frequencies we can see that in the layer with fixed surfaces the structural and nearsurface nonhomogeneities effect on the frequencies ν_n is negligible. In the layer with both free surfaces this effect is twice of the effect in the case of one surface free. These conclusions are similar to the results obtained in [9].

4. Conclusions

The key equation system of the locally nonhomogeneous solid is nonlinear due to nonlinearity of the momentum of mechanical translational motion. The model relations for describing such body take into account the structural nonhomogeneity of the body and the geometric nonuniformity of its surface, to coordinate the reference and actual states the mass source is used. The analysis of oscillating processes in the body under periodic external action can be carried out using the technique of averaging on the period of oscillation. Within the considered model the expression for frequency of natural oscillations includes two characteristic sizes, one of which is associated with the structural nonhomogeneity of the body material and another is related to geometrical nonhomogeneity of its surface. The frequencies of natural oscillations of locally nonhomogeneous layer depend on its thickness showing the size effect of magnitude depending on the fixing conditions of the layer surfaces. In the layer with both free surfaces the size effect of the natural oscillation frequencies is twice of such effect if just one surface is free. If both the surfaces of layer are fixed then the size effect is small.

[1] Eringen A. C. Nonlocal Continuum Field Theories. Springer, New York (2002).

[2] Wang Q., Liew K. Application of nonlocal continuum mechanics to static analysis of micro-and nano-structures. Physics Letters A 363 (3) (2007) 236–242.

[3] Burak Y. I., Nagirnyi T. S. Mathematical modeling of local gradient processes in inertial thermomechanical systems. Int. Appl.Mech. **28**, 775 (1992).

[4] Nahirnyj T., Tchervinka K. Mathematical Modeling of Structural and Near-Surface Non-Homogeneities in Thermoelastic Thin Films. Int.J.Eng.Sci, **91**, 49–62 (2015).

[5] Nahirnyj T., Tchervinka K. Thermodynamic models and methods of thermomechanics with regard to the nearsurface and structural non-homogeneities. Bases of nanomechanics I. Lviv, Spolom, 2012. – 264 p.

[6] Nahirnyj T., Tchervinka K. Basics of mechanics of local non-homogeneous elastic bodies. Bases of nanomechanics II. Lviv, Rastr-7, 2014. – 168 p.

[7] Mytropolskyj Yu. A. Averaging method in nonlinear mechanics. (1971).

[8] Hrebennikov E. A. Averaging method in applied problems. (1986).

[9] Nahirnyj T., Tchervinka K. Modeling of wave processes in solids with regard to the effects of nearsurface nonhomogeneity. Visn. Lviv un-ty. Ser. mech.-mat. – 1999. – Iss.54. – P.117-124.

[10] Nahirnyj T., Tchervinka K. Modeling and study of the temperature effect on natural oscillations of the layer. Visn. Lviv un-ty. Ser. mech.-mat. – 2002. – Iss.60. – P.102-106.

Хвильові процеси у локально-неоднорідних тілах

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Запропоновано метод вивчення хвильових процесів у локально неоднорідних тілах із врахуванням геометричної неоднорідності поверхні. Метод базується на системі рівнянь моделі локально неоднорідного пружного тіла, отриманій у межах локально градієнтного підходу, та використанні операції осереднення для розділення коливної та повільно змінної на періоді коливань складових полів переміщення та густини. На прикладі шару проілюстровано застосування методу до вивчення частот власних коливань для різних умов закріплення поверхонь шару. Встановлено, що залежність частот власних коливань шару від характерних розмірів приповерхневої та структурної неоднорідностей у випадку шару із вільними поверхнями є значно більшою порівняно із шаром, поверхні якого защемлені.

Ключові слова: *локально градієнтний підхід, приповерхнева та структурна неоднорідність, розмірний ефект, власні коливання*

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